Neil Turok. ©2025 Brett Eloff

Throughout the meeting, there were presentations representing key stakeholders in the region to help inform discussions between the societies. These included the Institute for Mathematical Sciences and its Applications (IMSA) (a partnership between a Centre of Excellence and a National Graduate Academy for mathematical and statistical sciences) the African Institute of Mathematical Sciences (AIMS) (nexteinstein.org) (a network of centres of excellence in mathematical sciences across Africa) and the South African National Research Foundation (NRF) (www.nrf.ac.za) (a national agency that supports research with funding and infrastructure). There was also a presentation from Sefako Makgatho Health Sciences University (SMU) (www.smu.ac.za), which outlined the challenges faced by historically disadvantaged institutions, namely universities that were marginalised during South Africa's apartheid era and which continue to serve communities previously denied access to higher education.

As the meeting progressed, there were several opportunities for attendees to discuss and propose areas of cooperation between the societies. These will be reported back to the LMS Global Affairs Committee and Council for consideration.

In addition to the three societies, the meeting received support from IMSA, the National Institute for Theoretical and Computational Sciences (NITheCS) (nithecs.ac.za), Pearson (za.pearson.com) and WITS University.

LMS Global Meetings: Future Plans

A third meeting will be held in Mumbai in November and will be reported on in a future issue of the *Newsletter*.

Report: Emily Riehl's Lecture at the University of Cambridge as part of the Hardy Lectureship Tour 2025

The final day of June 2025 reached 30°C, and its inwards-heat impact was felt strongly inside the Byzantine-inspired basilica of Churchill College Chapel. The afternoon started with a 20-minute piano performance by a doctoral student at Churchill College, Lena Alfter, who played one of Bach's canons (Canon in *Hypodiatessaron per Augmentationem in Contrario Motu*; BWV 1080, 14). The composition invited the audience to get prepared to immerse themselves in the lecture that was to be delivered by Professor Emily Riehl. This composition is composed of a canon voice that applies transformations to the original subject voice, where the transformations are of three types: interval reflection (an inversion or mirror up or down), augmentation (a stretch in time by a factor of 2) and transportation (a shift up or down by five semitones), symbolically summarised by: $g(t) = -f(t/2) + \Delta$ where f is the subject's voice, g is the canon's voice and Δ is equivalent to five negative semitones.

After listening to this composition, my mind drifted towards the affine groups: $\mathrm{Aff}(2,\mathbb{R}) = \{(A,b) : A \in \mathrm{GL}(2,\mathbb{R}), b \in \mathbb{R}^2\}$, and that is when the speaker, Professor Emily Riehl, who is on a UK tour from Baltimore, Maryland, began her talk on the topic *Path Induction and the Indiscernibility of Identicals*. The talk covered three main topics: dependent type theory, identity types and path induction and began with an introduction to induction over the natural numbers and finished with an epilogue on univalent foundations.

In the introductory section, while talking about induction over the natural numbers, Professor Riehl referenced Dedekind's 1888 book *Was sind und was sollen die Zahlen?* and Peano's 1889 paper 'Arithemetices principia, nova methodo exposita' to introduce

the axioms that characterise the natural numbers $\mathbb{N}$. An axiom introduced here is the *principle of mathematical induction*: $\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(\mathrm{suc}\, k)) \to (\forall n \in \mathbb{N}, P(n))$, wherein the triples in the set $\mathbb{N}$, an element $0 \in \mathbb{N}$ and the function $\mathrm{suc} : \mathbb{N} \to \mathbb{N}$ are isomorphic in nature. The variable P in this axiom predicates over the natural numbers. Given that the predicate is a function $P : \mathbb{N} \to \{\bot, \top\}$, one can prove that $\forall n \in \mathbb{N}$, $P(n)$ by proving the base case $P(0)$ and that the inductive step $P(k)$ implies $P(\mathrm{suc}\, k)$. An example of this that was shown is the proof that $n^2 + n$ is even, built over three steps: the base case, the inductive step and the principle of mathematical induction.

A constructive form of induction is the recursion in which the predicate $P : \mathbb{N} \to \{\bot, \top\}$ is replaced by an arbitrary family of sets $P : \mathbb{N} \to \mathrm{Set}$, such that the output of the recursive construction is a dependent function $p \in \prod_{n \in \mathbb{N}} P(n)$, which specifies a value $p(n) \in P(n)$ defined by p_0 and p_s satisfying the computation rules $p(0) := p_0$ and $p(\mathrm{suc}\, n) := p_s(n, p(n))$.

Like Peano's postulates, the things that characterise the natural numbers in dependent type theory are: type $\mathbb{N}$; element $0 : \mathbb{N}$; function $\mathrm{suc} : \mathbb{N} \to \mathbb{N}$; element $\mathbb{N}$-ind of type $P : N \to \mathrm{Type}$. The provable postulates not included in this list are that 0 is not a successor and suc is injective.

Professor Riehl then introduced dependent type theory. In this formal system for mathematical statements and proofs, the notions of types (e.g., $\mathbb{N}$, $\mathbb{R}$ and Group), elements (e.g., $17 : N$, $\sqrt{2} : \mathbb{R}$ and K_4 : Group), type families (e.g., $\mathbb{R}^- : N \to \mathrm{Type}$) and dependent functions (e.g., $\vec{0} : \prod_{n:\mathbb{N}} \mathbb{R}^n$) are primitively used. The type constructors used to build new types from given ones come with the following rules: formation, introduction, elimination and computation. The product types are governed by the following rules: ^{X}-form, ^{X}-intro and ^{X}-elim and some computation rules that relate pairings and projections. By contrast, function types are governed by the rules: $^{\to}$-form, $^{\to}$-intro and $^{\to}$-elim and some computation rules that relate λ abstractions and evaluations.

The step towards proving a mathematical proposition in dependent type theory is that of constructing an element in the type that encodes the following statement: a proposition of the type A and B with the modus ponens: $(A \times (A \to B)) \to B$ and the following recipe for the construction: $\lambda p.\, \mathrm{pr}_2\, p(\mathrm{pr}_1\, p) : (A \times (A \to B)) \to B$.

Professor Riehl then introduced the identity types. In the first order of logic, the binary relation = is governed by reflexivity ($\forall x, x = x$) and the indiscernibility of identicals ($\forall x, y, x = y$ implies that for all predicates P, $P(x) \leftrightarrow P(y)$), which can also be used to prove symmetry and transitivity. The identity types can be interpreted using the formation and the introduction rules: $^=$-form and $^=$-intro, which can be iterated as follows: given $x, y : A$ and $p, q : x =_A y$, there is a type $p =_{x=_A y} q$. This type does not always have an element because from the existence of homotopical models of dependent type theory, types are interpreted as 'spaces', elements are interpreted as points, element p is a path from x to y in A and the element h is the homotopy between the paths. The iterated identity types have higher structures.

The Martin-Lof's rules were then introduced for the identity types in full using $=_{_}$ form: and $=_{_}$ intro:, such that the elimination rule for the identity type defines an induction principle that is analogous to recursion over the natural numbers by providing sufficient conditions to define a dependent function out of the identity type family. Considering elements p as paths, $=_{_}$ elim is defined as the path induction for which the type family $P(x, y, p)$ over $x, y : A, p : x =_A y$ proves $P(x, y, p)$ with the following definition: path-ind: $(\prod_{x:A} P(x, x, \mathrm{refl}_x)) \to (\prod_{x,y:A} \prod_{p:x=_A y} P(x, y, p))$.

The next topic covered by Professor Riehl was path induction. The propositions for path reversion and path concatenation and the corresponding constructions are defined for the type family $P(x, y, p)$ over $x, y : A, p : x =_A y$. The identity types can be iterated with the given $x, y : A$ and $p, q : x =_A y$, with the type $p = x_{=_A y} q$.

The theorem by Lumsdaine, Garner and van den Berg says that the elements belonging to the iterated identity types of any type A forms an ∞-groupoid. The structure for this groupoid has elements $x : A$

as objects, paths $p : x =_A y$ as 1-morphisms and paths of paths $h : p =_{x=_A y} q$ as 2-morphisms.

The required structures proven by path induction are the following: constant paths (reflexivity), reversal (symmetry) and concatenation (transitivity), where concatenation is associative and unital with coherent associators. Path induction is helpful for proving the higher coherences in the ∞-groupoid of paths with the proposition that for any type A and elements $w, x, y, z : A$, the statement

$$\text{assoc} : \prod_{p:w=_A x} \prod_{q:x=_A y} \prod_{r:y=_A z} (p*q)*r =_{w=_A z} p*(q*r)$$

reaches the ∞-groupoid of paths.

Another proposition and construction used to understand higher coherences in the path algebra are the proposition $\mathrm{tr}_{P,p} : P(x) \to P(y)$ and the construction using the identity function $\lambda x.x : P(x) \to P(x)$. The corollary derived from this is that if $p : x =_A y$, then $P(x) \simeq P(y)$ using the construction that by path induction, assuming y is x and p is refl_x, we obtain an identity equivalence.

The end of the talk included an epilogue on the univalent foundations. Professor Riehl introduced a comprehensive table, like the Rosetta stone, that translates mathematical terms between the fields of type theory, logic, set theory and homotopy theory. Homotopical type theory suggests new definitions where the type A is contractible with an element of the type $\text{is-contr}(A) := \sum_{a:A} \prod_{x:A} a =_A x$.

This was followed by Voevodsky's hierarchy of types wherein a type A can be a proposition, a set or 0-type, or a suc$n$-type, provided that some conditions are met. An equivalence between types A and B gives an element that provides functions and homotopies. A universe $\mathcal{U}$ of types between small types A and B as its elements is given by $A, B : \mathcal{U}$ such that the univalence axiom is that the function $\text{id-to-equiv} : (A =_{\mathcal{U}} B) \to (A \simeq B)$ is an equivalence. Numerous consequences arise from this axiom, a few of which are the structure-identity principle, function extensionality and the indiscernibility of identicals. In fact, using path induction, it is possible to use Voevodsky's univalence axiom to apply the proof for one object to another that is equivalent to it. The justification of the path induction principle relies on the assertion that the mapping out of the path space is sufficed by defining the images of the reflexivity paths, such that the function: $\lambda x.(x, x, \mathrm{refl}_x) : A \to (\sum_{x,y:A} x =_A y)$ is an equivalence.

The lecture finished with Professor Riehl drawing references to the book *Homotopy Type Theory: Univalent Foundations of Mathematics* (homotopytypetheory.org/book) and the *HoTTEST Summer School* (discord.gg/tkhJ9zCGs9).

Saksham Sharma
Meeting attendee

Report: Insights from the LMS Hardy Lectureship 2025 at Cardiff University

C.P. Snow recounted how G.H. Hardy would work from 9am to 1pm on mathematics before retiring to the cricket pitch for the afternoon. The hot afternoon of 27 June was not one for lazing around in the sun at the School of Mathematics. The ground floor teemed with open day visitors, while upstairs Professor Emly Riehl (math.jhu.edu/~eriehl), this year's LMS Hardy Lecturer, inspired as a girl by Hardy's poetic description of mathematics as well as his admirable work–life balance, faced an excited crowd ranging from undergraduates to senior professors to expound on *Homotopy Types as Homotopy Types*.

Riehl arrived to us from a foreign land — not just from Johns Hopkins University in Baltimore but from an entirely alternative set of mathematical foundations. Homotopy type theory (often abbreviated as HoTT) builds mathematics on a foundation of types rather than sets, with abstract homotopy theory being used to understand these types. The first 'homotopy types' in the cryptic title, Riehl eventually confessed, referred to the objects of homotopy type theory. We were to be introduced to these through the more traditional 'homotopy types' of topology. Of all the talks Riehl would deliver on her tour through the UK, this one, we were assured, was the wackiest.

In the category of simplicial sets, the Kan complexes, the fibrant objects that are so fundamental in homotopy theory, are recast as 'spaces'. Fibrations became 'families of spaces'. For a space *A*, the central space for this talk was to be the path space of *A*, but in a simplicial way: the set of maps from the standard 1-simplex into *A*.

The principle of 'path induction' from homotopy type theory, Riehl explained, was that, for a family of spaces over a path space, any section defined on the constant paths could be extended to a section