Commentary on “Path Induction and the Indiscernibility of Identicals”, Hardy Lecture 2025

The final day of June 2025 reached 30° C, and its inwards-heat impact was felt strongly inside the Byzantine-inspired basilica of the Churchill College Chapel. The afternoon started with a 20-minute piano performance by a doctoral student at Churchill college Lena Alfter. Lena played Bach’s Canon (Canon in Hypodiatessaron per Augmentationem in Contrario Motu; BWV 1080, 14) for about 20 minutes. The composition invited the audience to get prepared to immerse in the esteemed Hardy lecture that was to be delivered by Prof. Emily Riehl. The composition under discussion is composed of a canon voice, that applies *transformations* to the original subject voice, where the transformations are of three types: interval reflection (inversion or mirror up/down); augmentation (stretch in time by a factor of 2); and transportation (shift up/down by 5 semitones), symbolically summarised by: $g(t) = -f(t/2) + \Delta$ where f is the subject’s voice, g is the canon’s voice, and Δ is equivalent to five negative semitones. Upon listening to this composition, my mind drifted towards the affine groups: $\text{Aff}(2, \mathbb{R}) = \{(A, b) : A \in GL(2, \mathbb{R}), b \in \mathbb{R}^2\}$, and that is when the speaker Prof. Emily Riehl, who is on a UK tour from Baltimore, Maryland began her talk on the topic “Path Induction and the Indiscernability of Identicals”. The talk covered three main topics: dependent type theory, identity types, and path induction, and was started by an introduction on “induction over the natural numbers” and finished with an epilogue on the “univalent foundations”.

In the introductory section, while talking about induction over the natural numbers, Prof. Riehl refer-

Figure 1: Stills from the Hardy Lecture 2025 at Churchill college chapel, Cambridge. Photo credits: Enrico Pajer and Benedikt Lowe.

enced Dedekind’s 1888 book “Was sind und was sollen die Zuhlen” and Peano’s 1889 paper “Arithematics principia, nova methoda exposita” to introduce the axioms that characterise the natural numbers $\mathbb{N}$. An axiom introduced here is the *principle of mathematical induction:* $\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(suck)) \to (\forall n \in \mathbb{N}, P(n))$, wherein the triples in the set $\mathbb{N}$, an element $0 \in \mathbb{N}$, and the function $\text{suc} : \mathbb{N} \to \mathbb{N}$ are isomorphic in nature. The variable P in this axiom predicates over the natural numbers. Given that the predicate is a function $P : \mathbb{N} \to \{\bot, \top\}$, one can prove $\forall n \in \mathbb{N}, P(n)$ by proving the base case $P(0)$ and the inductive step $P(k)$ implies $P(suck)$. An example of this that was shown is the proof that $n^2 + n$ is even, built over three steps: base case, inductive step, and the principle of mathematical induction. A constructive form of induction is the recursion in which the predicate $P : \mathbb{N} \to \{\bot, \top\}$ is replaced by an arbitrary family of sets $P : \mathbb{N} \to \text{Set}$, such that the output of the recursive construction is a dependent function $p \in \Pi_{n \in \mathbb{N}} P(n)$ which specifies a value $p(n) \in P(n)$ defined by p_0 and p_s satisfying the computation rules: $p(0) := p_0$ and

$p(\text{suc}n) := p_s(n, p(n))$. Similar to Peano's postulates, the things that characterise the natural numbers in the dependent type theory are: type $\mathbb{N}$; element $0 : \mathbb{N}$; function suc:$\mathbb{N} \to \mathbb{N}$; element $\mathbb{N} - ind$ of type $P : \mathbb{N} \to$ Type. The provable postulates not included in this list are: 0 is not a successor and suc is injective.

Prof. Riehl then introduced the dependent type theory. In this formal system for mathematical statements and proofs, the notions of types (e.g. $\mathbb{N}$, $\mathbb{R}$, Group), elements (e.g. $17 : \mathbb{N}$, $\sqrt{2} : \mathbb{R}$, K_4 : Group), type families (e.g.: $\mathbb{R}^- : \mathbb{N} \to Type$), and dependent functions (e.g. $\overrightarrow{O} : \Pi_{n:\mathbb{N}\mathbb{R}^n}$) are primitively used. The type constructors used to build new types from given ones comes with the following rules: formation, introduction, elimination, and computation. The product types are governed by the following rules: $^X-$form, $^X-$intro, $^X-$elim, and some computation rules that relate pairings and projections. On the contrary, function types are governed by the rules: $^\to-$form, $^\to-$intro, $^\to-$elim , and some computation rules that relate $\lambda-$ abstractions and evaluations. The step towards proving a mathematical proposition in dependent type theory is that of constructing an element in the type that encodes the following statement: a proposition of the type A and B with the modus-ponens: $(A \times (A \to B)) \to B$ and the following recipe for the construction: $\lambda p.\text{pr}_2 p(\text{pr}_1 p) : (A \times (A \to B)) \to B$.

Prof. Riehl then introduced the identity types. In first order of logic, the binary relation "=" is governed by reflexivity ($\forall x, x = x$) and the indiscernibility of identicals ($\forall x, y$ x=y implies that for all predicates P, $P(x) \leftrightarrow P(y)$) which can also be used to prove symmetry and transitivity. The identity types can be interpreted using the formation and the introduction rules: $^=-$form and $^=-$intro which can be iterated as: given $x, y : A$ and $p, q : x =_A y$, there is a type $p =_{x=_A y} q$. This type does not always have an element , because from the existence of homotopical models of dependent type theory, types are interpreted as "spaces", elements are interpreted as points, element p is a path from x to y in A, and the element h is the homotopy between the paths. The iterated identity types have higher structures. The Martin-Lof's rules are then introduced for the identity types in full using $=_{\_}$ form: and $=_{\_}$ intro:, such that the elimination rule for the identity type defines an induction principle which is analogous to recursion over the natural numbers by providing sufficient conditions to define a dependent function out of the identity type family. Considering elements p as paths, $=_{\_}$ elim is defined as the path induction for which the type family $P(x, y, p)$ over $x, y : A, p : x =_A y$ proves $P(x, y, p)$ with the following definition: path-ind: $(\Pi_{x:A} P(x, x, refl_x)) \to \left(\Pi_{x,y:A}\Pi_{p:x=_A y P(x,y,p)}\right)$.

The next topic covered by Prof. Riehl is the path induction. The proposition for the path reversion and the path concatenation and the corresponding constructions for this is defined for the type family $P(x, y, p)$ over $x, y : A, p : x =_A y$. The iteration of the identity types can be iterated with the given $x, y : A$ and $p, q : x =_A y$, with the type $p =_{x=_{A y}} q$. The theorem by Lumsdaine, Garner-van den Berg says that the elements belonging to the iterated identity types of any type A forms an $\infty-$groupoid. The structure for this groupoid has elements $x : A$ as objects, paths $p : x =_A y$ as 1-morphisms, paths of paths $h : p =_{x=_A y}$ q as 2-morphisms. The required structures proven from the path induction principle are the following: constant paths (reflexivity), reversal (symmetry), concatenation (transitivity), where the concatenation is associative and unital with coherent associators. The path induction is helpful to prove the higher coherences in the $\infty-$ groupoid of paths with proposition that for any type A and elements $w, x, y, z : A$, the statement assoc: $\Pi_{p:w=_A x}\Pi_{q:x=_A y}\Pi_{r:y=_A z(p*q)*r=_{w=_A z}p*(q*r)}$ to reach up to the $\infty-$ groupoid of paths. More propositions and constructions to understand higher coherences in the path algebra is the proposition $tr_{P,p} : P(x) \to P(y)$ and construction using the identity function $\lambda x.x : P(x) \to P(x)$. The corollary derived from this is that if $p : x =_A y$, then $P(x) \simeq P(y)$ using the construction that by the path induction, assumning y is x and p is refl_x, we obtain an identity equivalence.

The talk reached towards the end with an epilogue on the univalent foundations. Prof. Riehl introduced a comprehensive table on the Rosetta stone that translates mathematical terms between the fields of type theory, logic, set theory, and homotopy theory. The homotopical type theory suggests new definitions where the type A is contractible with element of the type is-contr(A):= $\Sigma_{a:A}\Pi_{x:A} a =_A x$. This was followed by Voevkodsky's hierarchy of types wherein a type A can be a proposition, a set or 0-type, or a sucn-type, provided that some conditions are met. An equivalence between types A and B gives an element which provides functions and homotopies. A universe $\mathcal{U}$ of types between small types A and B as its elements is given by $A, B : \mathcal{U}$ such that the univalence axiom is given by the function id-to-equiv : $(A =_{\mathcal{U}} B) \to (A \simeq B)$ is an equivalence. A numerous consequences arise out of this axiom, a few of which are: structure-identity principle; function extensionality; and the indiscernability of identicals. In fact, using path induction, it is possible to use Voevodsky's univalence axiom to apply proof of one object to another which is equivalent to it. The justification of the path induction principle relies on the assertion that the mapping out of the path space is sufficed by defining the images of the reflexivity paths, such that the function: $\lambda x.(x, x, refl_x) : A \to (\Sigma_{x,y:A} x =_A y)$ is an equivalence. The lecture finished by Prof. Riehl drawing references to Homotopy Type Theory: Univalent Foundations of Mathematics (link) and the HoTTEST summer school (link).