To Challenger (21/12/24):

Following the discussion yesterday evening, I went back to the complexity foundations of Sarnak conjecture. As elaborated by P. Sarnak in YT (May 27, '21) talk / link sent by e-mail), the Vinogradov's Mobius Randomness Heuristic could serve as a good analogy for the 'arguments' below. My goal is to elucidate how the disjointness / non-correlation between Mobius function $\mu(n)$ and bounded sequence $\xi(n)$ could be directly / indirectly (?) transferred to the "10^4 in 10^4" and "1 in 10^4 error" functions [let's call them Truth and Error functions, respectively] and any "potential" disjointness between them would be a nice conjecture or theorem to have.

In Sarnak's analysis, there are two parties P_1 and P_2. Two functions $\mu(n)$ and $\xi(n)$ are at play. Each of these functions can have, vaguely, low or high (computational) complexity. The rule or law that is at check here is the following heuristic

$$\frac{1}{N}\sum_{n\leq N}\mu(n)\xi(n)\to 0, \text{ as } N\to\infty$$

for any bounded sequence $\xi(n)$. The two parties, each, have three potential 'state of mind': belief, a mathematical statement (conjecture / theorem), question. Let us denote it, respectively, by B, M, Q. The following table is the state-of-the-art of the topic:

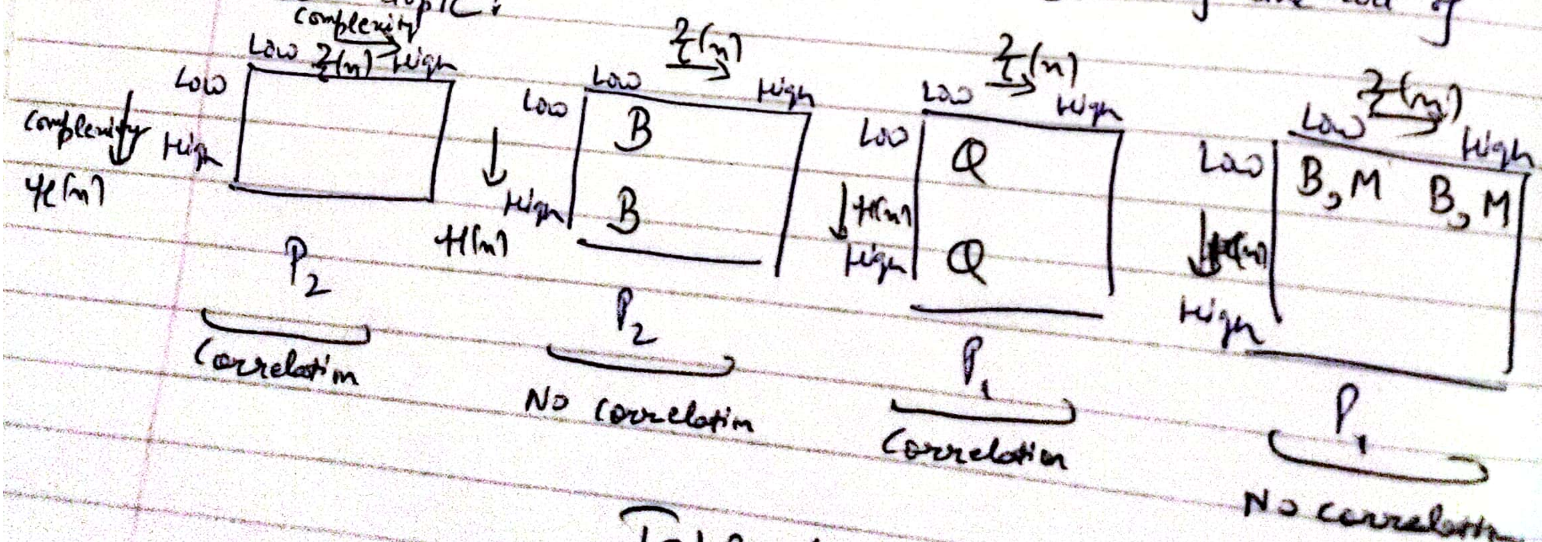


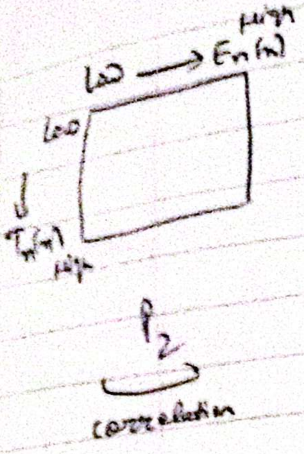
Table 1

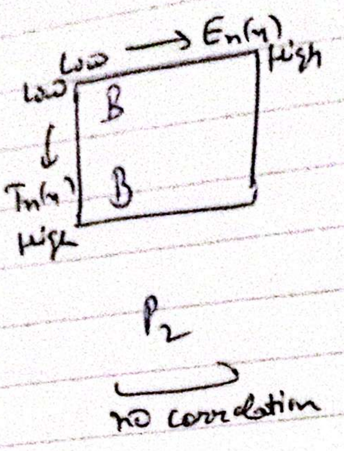
The reasoning that comes out of table 1 is that P_2 believes $\xi(n)$ must be a low complex bounded sequence in order for "no correlation" with $\mu(n)$ to hold true. It's a safe guess to make, especially when one thinks that factoring is hard in complexity, and any $\xi(n)$ that doesn't correlation with hard-$\mu(n)$ must be of a low complexity nature. However, P_2 can easily be assessed on this belief by P_1 when P_1 asks: Can there exist a low complex $\xi(n)$ correlated with $\mu(n)$? An answer to such a question would be of two forms: yes ~~or~~ no. In case of yes, P_2 contradicts themself. In case of no, P_2 would be able to support their belief (maybe partially). For P_1, there is a lot more here. With an added belief that $\mu(n)$ is of low complexity, if P_1 comes up with a conjecture (or theorem) that $\mu(n)$ and $\xi(n)$ are not correlated, there is a shift from belief B and question Q in this 2-player game, to a rather concrete Statement M for [illegible] (easy-$\mu(n)$, easy-$\xi(n)$) and (easy-$\mu(n)$, hard-$\xi(n)$) cases. The former is where both players are interested in and would agree to appreciate the result, and the latter is where further explorations on high complexity $\xi(n)$ would open up.

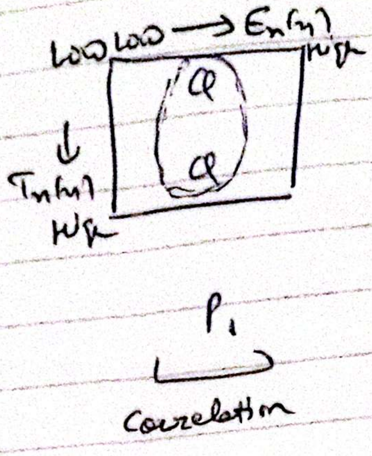
Let us try to transfer such a game to our discussion on "10^4 in 10^4 cases" (Truth) and "1 in 10^4 cases" (Error) functions. Let us denote by T_n and E_n respectively. The question/heuristic to test is if:

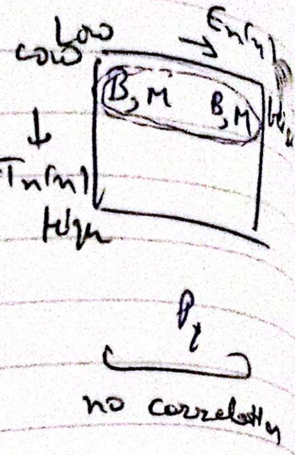
$$\frac{1}{N}\sum_{n\le N} T_n(n)\, E_n(n) \to 0, \text{ as } N\to\infty$$

where $E_n(n)$ is an error sequence sampled from a given probability distribution function, say $N(1,1)$. In the next page, I attempt to draw the same table as before, adapted to the present case of string theory & AI project (details are unknown to me):









The game becomes interesting in this case. P_2 believes (B) that only a low-complex error function (E_n) [trivially, zero error] would yield an error-free (10^4 in 10^4) case truth function [aka, classical apriori algorithm of P-time]. P_1 questions this belief by asking "Can you disprove the counter-opposite, by finding an arbitrary-complex error function [sample from any of the "permissible" p.d.f.], because if you do it, I'll stop working and not find any theorem, as your belief seems more valid to be correct. However, if you can't, I will believe that a low-complexity truth function ($T_n(n)$) exists and go on to test all the possible samples (p.d.f.) of error functions $E_n(n)$ from the p.d.f. space, to form a cojecture that correlation b/w truth function and error function sampled in the p.d.f. space is zero. Perhaps, to formulate such a cojecture, translation of $E_n(n)$ to a mesh of rational numbers which model various samples and also use monotonicity properties of functions to precise control errors (by mapping back), could be a nice idea. In effect, I think one needs to translate the error function to a discrete space which could be more 'rigorously' controlled, s.t., a disjointness/non-correlation condition to the truth function $T_n(n)$ [which is unknown to us in complexity sense] could be achieved.

vague & exploratory for now