Reeb field $\to$ $\vec{F}$ on $C[\mathbb{R}^3]$.

Hopf field

Hopf algebra:

$a \otimes b = b \otimes a$ $\leftarrow$ Hopf algebra

$$H^i(kG, A) \simeq H^i(G, A^*)$$

$\mathrm{Hom}(C, A)$

If A is an UL-module, then for $i \geq 1$,

$$H^i(UL, A) \simeq H^i(L, A^+)$$

Sheaves

Network, Complex, Cell, Site Topology, Site

Topographs, my case

(1) Cellular sheaves:

$$C^0(f) \longrightarrow C^1(f) \longrightarrow C^2(f)$$

Vertex data, Edge data, 2-cell data, 3-cell data.

Graph Laplacian

$$H^0(f) \longrightarrow H^1(f)$$

Hodge Laplacian

For X a graph, f the constant sheaf, L^0 is the graph Laplacian

Hodge theorem: $H^k f = \ker L^k$.

(2) Tarski Laplacians

Tarski cohomology: $H^k f = (T_d^k f)$

$\to$ Erdős Problem (#735) Ben Morrell

Stationary solns of Euler eqns

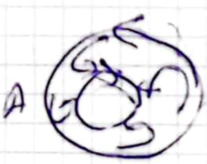
of non-vanishing vector fields $X \neq 0$ on $M = S^3$/closed

Hydrodynamical	Symplectic	Regularity	Always $\exists$ periodic orbits
1) arbitrary $\downarrow$ Imposing geometric constraint $\nexists$ UH (div free)		C^1 C^ω	No Schweitzer '74 No (K. Kuperberg '94).
2) Volume preserving $L_X \mathrm{vol} = 0$ $\parallel$ $d(\underbrace{i_X \mathrm{vol}}_{=:\omega})$ UH (div free)	Hamiltonian $\omega \neq 0$ closed $L_X \omega = 0$		No (G... '98).
3) Euler $\nabla_X X = -\nabla p$ $\alpha = g(X, \cdot)$ $h = p + \frac{1}{2}\lvert X\rvert^2$	$\Longleftrightarrow$ $i_X d\alpha = -dh$ $\alpha(X) > 0$.	C^ω (Geiges) C^∞	Yes (Etnyre-Ghrist '00). ?
	(g,α) Stable Hamiltonian structure		
Beltrami fields: $\exists$	$\longleftrightarrow$ $i_X d\alpha = 0$, $\alpha(X) > 0$		Yes Hutchings–Taubes '09.
curl $X = X$ UH	$\longleftrightarrow$ α contact X Reeb	C^∞	C^2 (Yes) Hofer '93 on S^3 Taubes '07 on M^3

Obstruction to Euler: Reeb comp.

(Like α-plug)

$$0 = \int_{A_{\text{Can}}} d\alpha = \int_{\partial A} \alpha > 0 \ ↯$$



One can change any vol-pres. X by inserting a plug with a Reeb component keeping it v.p. pres.

Assume:

$i_X d\alpha = -dh$

$\alpha(X) \geq 0$

$\Rightarrow h$ integral of motion

$\rightarrow h =$ constant.

$\rightarrow i_X d\alpha = 0$.

(2) Given X: $i_X d\lambda = -dh$

$\lambda(X) > 0$

Want γ: $i_X d\gamma = 0$

$\gamma(X) > 0$

* Bernoulli motion (fluid): Result &

Bernoulli function → (Perelt-Sols-poly) "

Idea: Assume h is C^ω

Key story!

Closed 1-form: ($d\gamma = 0$)

Construct "γ" near critical level sets of "h".

* Integrals are like degrees of freedom!

Away from crit. level sets, foliation by invariant tori,

$I \times T^2$ (...) flow is linear.

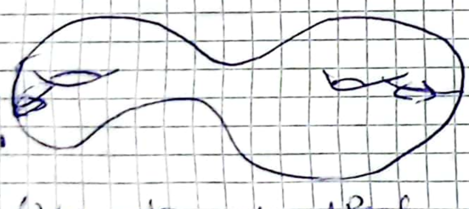
Why does it not work in smooth case? (counterexamples in C^∞ case)

[* Look at all other counterexamples related to smooth solutions]

$\Sigma = H/\Gamma$ — discrete subgroup SL^2 — closed hyp. surface

canonical contact form

$(S^*\Sigma, \lambda)$ contact, Reeb vector field R



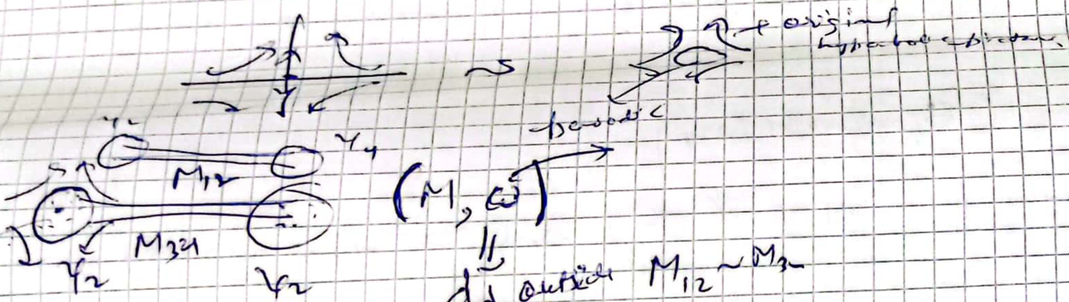
$\bar\gamma_3 = -\bar\gamma_1$ $\bar\gamma_2 = -\bar\gamma_4$

Lift to closed Reeb orbits $\{\gamma_1, \dots, \gamma_4\}$

$\gamma_1 \sim -\gamma_3$ $\gamma_2 \sim -\gamma_4$

$$\int_{\gamma_1}\lambda = \int_{\gamma_3}\lambda = T_1 \neq T_2 = \int_{\gamma_2}\lambda = \int_{\gamma_4}\lambda$$

Modify d on orbifold V_i of γ_i, to d contact



(M, ω)

$\Downarrow$

d_d outside $M_{12} \sim M_{34}$

Hard to understand. But these are the self, where we are leaving the analytic category!

Lemma 1 $\exists X$ s.t. it is Euler

Lemma 2 $\exists \tilde{X}$ s.t. it is not Euler

($\Rightarrow$ not Beltrami)

proof

Q If every rescaling of X is Euler,

(Tao) Is X also Beltrami?

Take $X \sim \tilde{R}$ outside $M_{12} \cup M_{34} = (3,4) \times T^2$

$\qquad \qquad \qquad \qquad \qquad \qquad \| [1,2] \times T^2$

s.t. $X = \partial_2$ $\quad r \longrightarrow \varphi, z$

on $\{i\} \times T^2$.

$\Rightarrow \tilde{X}$ does not extend over $M_{12} \cup M_{34}$ as Euler: If $X^0 \sim \tilde{X}^0 \cdot \mathcal{I} + \beta$

$\gamma(\tilde{X}) > 0$ $\qquad \gamma(X) > 0$

$\gamma(\tilde{X}) > 0$; $i_{\tilde{X}} d\nu = -dh$, $\quad i_{\tilde{X}} d\nu = -dh$

$$0 = \int_{\gamma_2} \nu - \int_{\gamma_1} \nu = c(T_2 - T_1) + \langle \beta, [\gamma_2] - [\gamma_1] \rangle.$$

$$0 = \int_{\gamma_4}^{\gamma_2} \nu - \int_{\gamma_3}^{\gamma_1} \nu = c(T_2 - T_1) + \langle \beta, [\gamma_4] - [\gamma_3] \rangle$$

$$0 = 2c(T_2 - T_1) + \langle \beta, \overbrace{[\gamma_2] + [\gamma_4]}^{=0} - \underbrace{([\gamma_1] + [\gamma_3])}_{=0} \rangle \}.$$

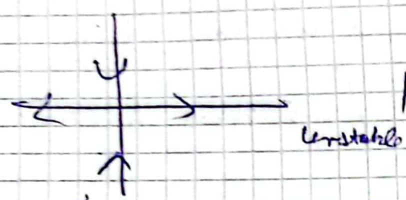
Conj (Birkhoff 194y)

Let $f \in \mathrm{Symp}^{\omega}(S^2)$ s.t. $\#\,\mathrm{fix}\, f = \#\,\mathrm{Per}\, f = 2$.

Conj (Birkhoff 1927)

formal stability $\not\Rightarrow$ stability $\Leftarrow$ Conj.: There are unstable elliptic fixed pts.

$\circlearrowleft$

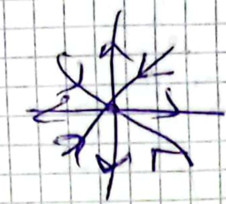


$H(x,y) = xy$

hyp = instability

$z \mapsto z^{q}$

(x,y)



$\tilde{H} = H(z^q)$,

$R_{p/q} \circ \phi_{\tilde{H}}$

maybe stable?

★ A fixed point is elliptic if its eigenvalues are in $S^1 = 1$ with irrational argument.

★ A fixed p is unstable if $\exists$ open $V \ni p$, $\forall$ open $U \ni p$, $\bigcup_n f^n(U) \not\subset V$

Theorem: Anosov – Katok 1970:

There are ergodic smooth symplecto of the cylinder or the sphere without periodic or two periodic points

Important slide

Theorem (Newlander – Nirenberg):

If $(J_n)_n$ is a sequence of complex structures on M_p, which C^∞-converges to J_∞, then J_∞ is complex structure, and if $(\Omega_n)_n$ is seq. of J_n-holomorphic extensions of Ω which converges to Ω_∞, then Ω_∞ is a J_∞-holomorphic extension of Ω.

Contradiction (or (nobyour of sequence) method for B

automorphism group (and related extensions) → see the slide in pfs of Mean approx. theorem [Pierre Berger]

3rd talk (Ana)

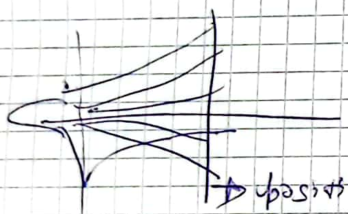
C^∞-generic Reeb flows & hyperbolicity,

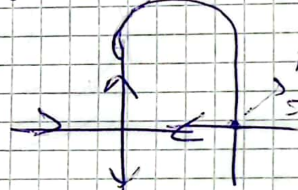
M-closed oriented 3-manifold (S^3)

Defn: A periodic orbit is non-degenerate if 1 is never an eigenvalue of DP^n, $\forall n$.

X is volume preserving, two types non-deg. of orbits

- hyperbolic (real EVs) ⇝ Unstable and stable mfds.
- elliptic → DP an irrational rotation.





⇒ positive topological entropy
⇒ infinitely many periodic orbits.

Thm: [CDHR]

C^∞-generic Reeb vector fields have homoclinic orbits for all hyperbolic periodic orbits.

Hypothesis: X is Reeb vector field – strongly non-degenerate
- periodic orbits are equidistributed ⇒ (homotopical motion?)

X is Reeb, there is a contact form 'α',

s.t., $\alpha(X)=1$, $L_X d\alpha = 0$

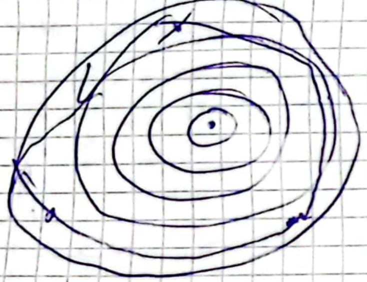
$\alpha \wedge d\alpha > 0$ is an invariant volume form.

C^∞-generic Reeb vector fields (Irie '23) pose ambedded (contact ~~topology~~ homology)

Check more about it later!

— Zehnder's condition

Near an elliptic periodic orbit



— Remark

Implies that hyperbolic periodic orbits are dense in M.

Theorem (Contreras, Olviera 22 (preprint))

(Σ, g) a Riemannian surface,

C^∞-generically on 'g' every hyperbolic periodic orbit of the geodesic flow has a transverse homoclinic.

Thm ((DHR, Fontreras – Mazzucchelli))

A C^∞-generic Reeb vector fields admits a Birkhoff section

Birkhoff section

(Contreras – Mazzucchelli) strongly non-degenerate

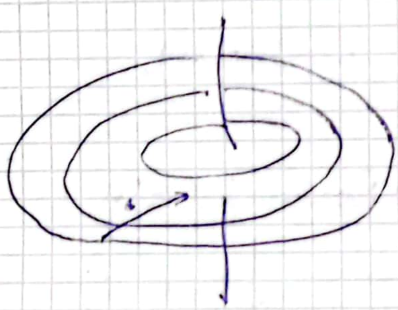
•(CDHR) non degenerate + equidistributed

A Birkhoff section

(M^3, X) a Birkhoff section [dynamical characterization of M]

$S \to M$ immersed surface

(δS) is a collection of periodic orbits of X.



The other ingredients:

Thm (le Calvez, Sambarino '23)

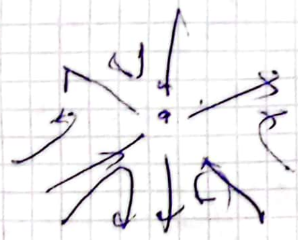
S is a closed surface

$f: S \to S$ an area preserving C^1-diffeomorphism

→ f non-degenerate

→ number of hyperbolic points $\geq \max(0, 2g-2)$

⇒ every hyperbolic orbit has a homoclinic.



Q A Reeb vector field like this without elliptic periodic orbits. How far is it from being Anosov?

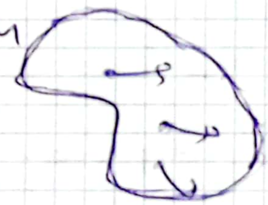
Corollary: of the main theorem

When the generic hypothesis, the Reeb vector field admits an embedded Birkhoff section (up to adding lots of boundary components).

→ Boris Khesin (Geodesics, vortex sheets, and generalised fluid flows)

$\partial_t v + \nabla_v v = -\nabla p$

$\operatorname{div} v = 0$; v is tangent to ∂M.



M — vol form μ

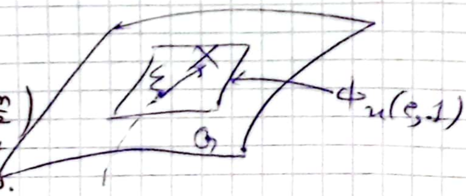
Theorem (Arnold 1966)

$G = \mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms w.r.t. the right-invariant L^2-metric

$$E(v) = \frac{1}{2}\int_M (v,v)_\mu$$

Suitable functional-analytic settings of Sobolev (H^s for $s > 1+\frac{n}{2}$) and some Fréchet (C^∞ spaces).

$C^\infty(S^1, SO(3)) \to H^1 \to$ Heisenberg magnetic chain.



G is normal of Γ complement of G.

Cordoba: (17/9/24)
→ Elgindi (2021)

Elgindi & Pasqué (2023) $C^{1,\alpha}$ (with swirl).

Placing bumps in outward / inward direction.

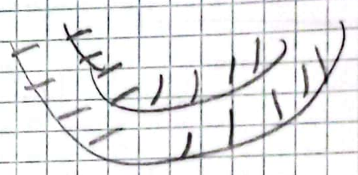
Axi-sym. Euler without swirl:

$$\omega_t + u^r\omega_r + u^z\omega_z = u^r\omega/r.$$

Step 1: Find a mech. for growth.

Infinite no. of bumps; each bump is comp. supp. at

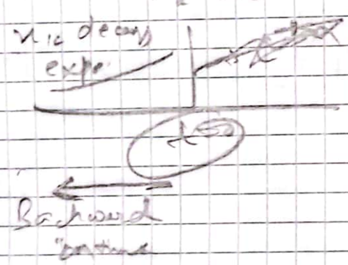
k^{th} bump:

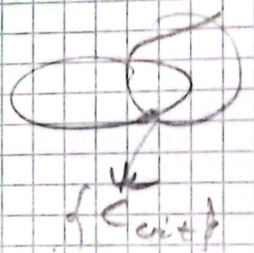


$$\omega = \omega_k + \sum_{j<k}\omega_j + \sum_{j>k}\omega_j.$$

$\Lambda_k \to \infty$

ODE:





D.C., L. Martinez-Zoroa (2023):

$\{C^{1,\frac{1}{2}-\varepsilon} \cap L^2\}$ for a

$$\omega \simeq A\,\psi(\lambda x_1)\,\psi(\lambda x_2)\,\psi(\lambda x_3)\sin(Mx_2)\sin(Mx_3)\,e_1$$

$+ \cdots + e_3$ (to make it div-free)

$$\omega = \sum_{n=0}^{\infty} \omega_n(x,t)$$

Hou:

Riesz transform:

$K: \nabla(\nabla u) = K\omega$.

Elgindi (2021): C^α vorticity, for small α.

$\exists$ Euler singularity for axisymmetric Euler

$(\omega_t + u\omega_x = u_x\omega$, if $\omega \sim cx^\alpha)$.

Hou-Luo, 08:

$$u_1 = \frac{u^\theta}{r}, \quad \omega_1 = \frac{\omega^\theta}{r}, \quad \psi_1 = \frac{\psi^\theta}{r}.$$

$$u_0^\theta(r,z) = r\,u_0(z), \quad \omega_0^\theta(r,z) = r\,\omega_0(z),$$

Lyapunov: $(u_{1,z})^2 + \omega_1^2$; and showed that

$$\left((u_{1,z})^2 + \omega_1^2\right)_t + (u^r, u^z)\cdot\nabla_{r,z}\left((u_{1,z})^2 + \omega_1^2\right) = 0$$

Th 1 (Chen-Hou, 2022, 2023):

1) Smooth initial data (θ_0, ω_0)

2) Comparison with Nonlinear Schrödinger eqn (~~analysis~~ explicit quantified).

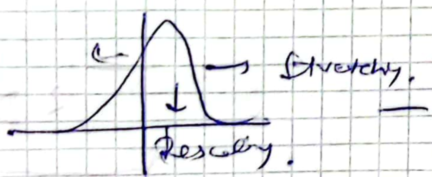
3) Computer assistance to obtain $\boxed{\text{'sharp'}}$ upper bounds for our stability constants.

→ look it up?

→ 2D Boussinesq ~~at~~ eqn. in $\mathbb{R}^2_+$:

→ 'stretching term' and 'scaling term'.

$c_l(\tau)x\cdot\nabla$ $\quad$ $c_\omega(\tau)\omega$



A general approach to nonlinear stab.:

$v = (\omega, 0)^T$; write

$$\frac{dv}{d\tau} = F(v)$$

and $\|F(\bar{v})\| = \epsilon \ll 1$.

$v = \bar{v} + \tilde{v}$; eqn. for $\tilde{v}$ is:

bilinear term

$$\frac{d\tilde{v}}{d\tau} = F(\bar{v} + \tilde{v}) = \underbrace{\nabla F(\bar{v})\tilde{v}}_{\text{linear}} + \underbrace{\tilde{v}^T \nabla^2 F(\bar{v})\tilde{v}}_{\text{nonlinear}} + F(\bar{v}).$$

$$\tilde{v} = \int_0^\tau e^{(\tau - s)L} \left(N(\tilde{v}) + F(\bar{v}) \right) ds.$$

Assume, $\|e^{\tau L}\| \le e^{-d\tau}$, $d > 0$.

and nonlinear term is bounded as:

$$\|N(\tilde{v})\| \le C\|\tilde{v}\|^2.$$

Then we have, $\|\tilde{v}(\tau)\| \le \int_0^\tau e^{-d(\tau - s)} \left(C\|\tilde{v}(s)\|^2 + \epsilon \right) ds.$

If $4\epsilon C < d^2$, $\exists\ \delta > 0$, s.t.

$$\frac{C\delta^2}{d} + \frac{\epsilon}{d} \le \delta.$$

Bootstrapping: Suppose $\|\tilde{v}(s)\| \le \delta$, $\forall\ s \in [0, \tau)$, then

$$\|\tilde{v}(\tau)\| \le \int_0^\tau e^{-d(\tau - s)} \left(C\delta^2 + \epsilon \right) ds$$

$$< \frac{C\delta^2}{d} + \frac{\epsilon}{d} < \delta$$

$\Rightarrow \|\tilde{v}(\tau)\| \le \delta$ $\forall$ time given $\|\tilde{v}(0)\| \le \delta$.

Ball

Not expl.

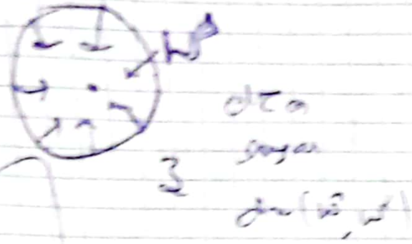
$\boxed{\text{3D NS E — explicit state (ground)}} \rightarrow \int_0^t B(u, u)\, dt$

Inspired from NLS.

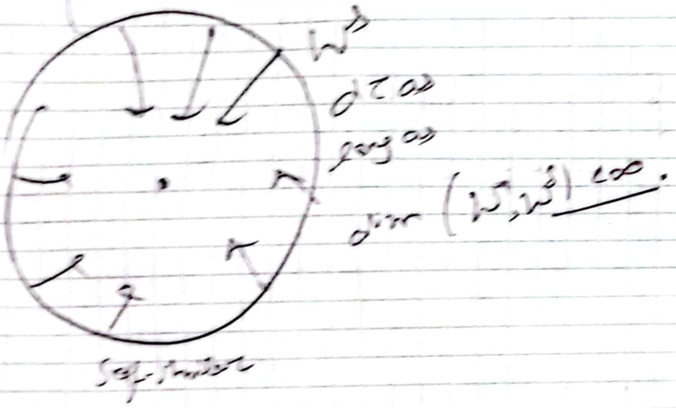
Look at the operator's regularity.

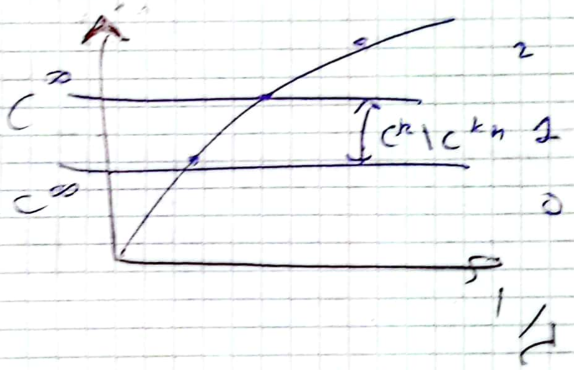
Program:

1. find a precise approximate (unstable) self-similar sol.
2. Decompose the $B(u_\lambda)$ into explicit part + finite dim. matrix + $\boxed{\text{a tail}}$ that can be understood by hand or bounded (using relative compactness).



3. Show that it is stable modulo a finite no. of directions assuming the smallness of large dir.
4. One can add some viscosity as long as it is exp. small in self-similar coordinates.
5. Prove non-linear stability.

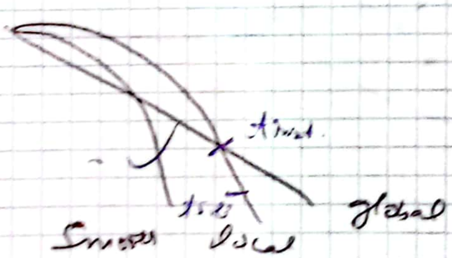




ODE System

The equation for w, Z reads

$$Z\,\partial_Z\, W = \frac{N_W(W,Z)}{D_W(W,Z)}, \quad Z\,D_Z\,Z =$$



$$\Omega = \text{curl}\, v = \delta_{y_1} v_2 - \delta_{y_2} v_1 \;;\; \phi = \delta_{y_1}\Theta \;;\; \psi = \delta_{y_2}\Theta .$$

Training:
1)
2)
3)

Cardinal (time-dep)

Helmholtz (thm): Given v_t, α, ..., then $\text{curl}\, v_t = (\psi_t)_* \text{curl}\, v_0$

where ψ_t $M \to M$ generating v_t ($D_t \vec{\psi} = v$)

Assume curl $u \neq 0$, → we fix its homotopy class "α".

Phenomena: $\mathcal{U}$ open set $= \pm\{ v \in X_\mu(M) \mid \mathcal{E}(u) = e,\; H(u) = h,\; \text{curl}\, u \in \alpha \}$

α, h, e
restrict helicity, value helicity ?
make more parameters here!

Thm (Khesin, ... 4): In $\mathcal{U}_{\alpha, h, e}$, $\forall h, \alpha$, $e \geq e_0(\alpha, h)$, the Euler equation is non-mixing with C^k-topology ($k \geq 6$)

Quest (Khesin-Misiolek-Shnirelman): lower k?

II. $\left[\frac{d}{dt} \text{ (contact hydrodynamics) } \right]$

Defn: Given $Y \in X^0_\mu(M)$, it is of contact type if $\iota_{Y}\mu = d\beta$ which is a contact primitive.

Prop.: $\{$ exact fields of contact type "carrying" $[\xi]\}$ is a $\mathcal{C}^0$-open set in $\mathfrak{X}^0_{\mu}(M)$.

Defin: An initial condition $v_0 \in \mathfrak{X}_{\mu}(M)$ is of contact type if curl u_0 is of contact type.

Rmk: If u_t is a soln to Euler, then v_t is of contact type $\forall t$.

$$\begin{cases} \operatorname{curl} u_t = (\varphi_t)_* \operatorname{curl} v_0 \\ \varphi_t^* \mu = \mu \end{cases} \Rightarrow \varphi_t^* d\beta_0 = d\beta_t, \text{ where}$$

$\beta_t = g(v_t, \cdot)$, so if α is contact and

$d\alpha = d\beta_0$

$\Rightarrow d(\varphi_t^* \alpha) = d\beta_t$.

Corollary:

$\mathcal{C}_{a,h,e}(\xi) = \mathcal{C}(\xi) \cap \mathcal{U}_{a,h,e}$ is $\mathcal{C}^1$-open in $\mathcal{U}_{a,h,e}$.

Remark: $\mathcal{C}_{a,h,e}(\xi)$ might be empty, so to make it non-empty

$\xi \oplus a = TM$.

Theorem (S Torres & Lázaro '23): $\mathcal{C}^1$-open sets

1. $\forall a, h \neq 0, e > c_0(a,h)$, $\exists\, U_1, U_2 \subset \mathcal{U}_{a,h,e}$, s.t. U_1 are contact type, U_2 are not contact type.
2. Under a technical assumption on a, given ξ s.t. $\xi \oplus a = TM$, $\exists$ functions $c: \mathcal{C}_{a,h,e}(\xi) \to \mathbb{R}$.

(f) Proposed quantity ... Continuous with C^0 topology !!! Non-trivial

$\forall e \in (a,h)$

Picture: $\mathcal{U}_{a,h,e}$ in CS^3, general!, $a = [\text{...}]$

is this for space a moduli space?

look later at it!

(picture of ... spaces)

Proof: Rank: C' is obtained from ECH.

1. Step 1: $\forall a, h \neq 0$, construct $\psi_1, \psi_2 \in \mathcal{X}^0(M, \ldots)$,

$\psi_1, \psi_2 \in a$, homotopic to ψ.

and ψ_1 is of contact type.

ψ is in set of contact type C^0 robustly.

look it up

(Giroux: obstruction for stability of orbits)

2. Step 2: $\forall c > e_0(a,h)$, find ψ_1, ψ_2, s.t. ... $\cong \psi_1$ (these?)

(i) Given a contact $\alpha \rightsquigarrow ECH(M, \alpha) \cong HM(M, 2)$

Hutchings, Taubes

$\exists\ \tilde{C} = C^0(H) \ldots$ for

$(f \longmapsto \tilde{C}(f \cdot \alpha))$

Properties:

(i) C^0-continuous

(ii) often depends only on $d(f\alpha)$

(iii) nontrivial

$C(w) := C(\alpha)$; for some primitive of $d\beta$

(iii) Abbondandolo – Bramham – Hryniewicz – Salomão

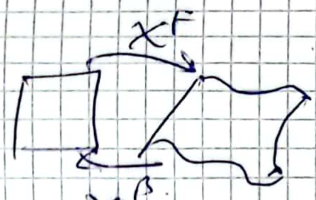
[nontrivial]

~~Dondollerausa~~:

Eulerian versus Lagrangian perspective (2D Euler).

flow map X

$X: D \times \mathbb{R}^+ \to D$



$X^B = (X^F)^{-1}$

$u: D \to \mathbb{R}^2$
$\downarrow$
domain.

$(\nabla \cdot u) = 0$

$\det(\nabla X) = 1$

$$\left.\begin{aligned} &\partial_t \varphi + (u \cdot \nabla)\varphi = 0 \\ &\varphi(x, t=0) = \varphi_0(x) \end{aligned}\right\} \text{ method of characteristics}$$

Looking for: $X^B(\gamma(t), t) = \gamma_0$, for all curves $\gamma(t)$, (pullback)?

Properties of X: $X_{[t_1,t_2]} \circ X_{[t_0,t_1]} = X_{[t_0,t_2]}$

$X(\cdot, t)$ is a one-parameter semi-group.

If $t_1 < t_2$, we have forward map.
if $t_1 > t_2$, backward map.

2D comp. Euler: $\partial_t u + (u \cdot \nabla) u = -\nabla p$.
$\nabla \cdot u = 0$
$u(x, t=0) = u_0(x)$,

$\boxed{\omega = \nabla \times u}$

$\partial_t \omega + (u \cdot \nabla)\omega = 0$

$\nabla \cdot u = 0$

Biot-Savart's law.

$\boxed{u = -\Delta^{-1}(\nabla \times \omega)}$

$\boxed{\nabla \omega_{(x,t)} = \nabla \omega_0 \cdot \nabla X(x,t)}$

$\nabla X_{[t,0]} = \nabla X_{[\tau_1, 0]} \, \nabla X_{[\tau_2, \tau_1]} \cdots$

Vortex stretching

3D Euler: $\left[\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u\right] + IC \ (\omega = \omega_0)$

Numerics: $(\partial_t u + u \cdot \nabla) x^{\beta} = \omega$

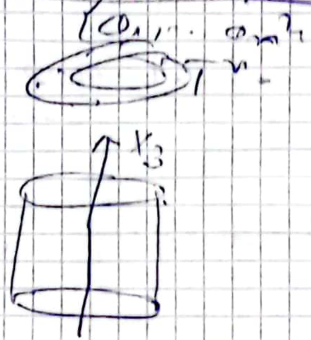
$$\{\omega(t,x) = (\nabla x^{\beta})^{-1} \omega_0(x^{\beta})\}$$

Where $u = -\Delta^{-1}(\nabla \times \omega)$, $\det(\nabla x^{\beta}) = 1$

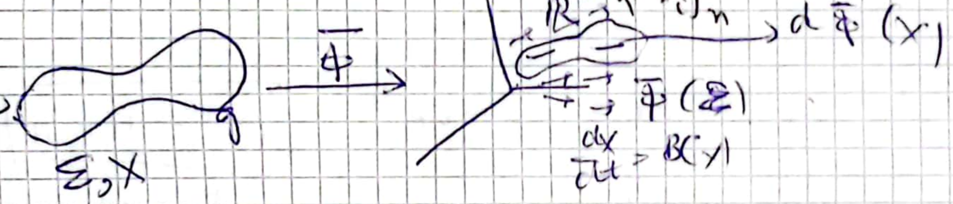
→ Numerics

→ $(\theta_1, \theta_2, \dots, \theta_m) \in T^m$

$$\frac{d\theta_i}{dt} = \omega_i$$



Proof:



$$B(x) = d_{\Phi^{-1}(x)} \Phi(x)$$

$$\boxed{\Delta \psi_i = \lambda_i \psi_i}$$

For large N, this is an embedding

Lemma: $\Phi(p) = (\psi_1(p), \psi_2(p), \dots, \psi_n(p))$

$$\Sigma \to \mathbb{R}^n.$$

~~Daniel Lemaride~~

Eulerian versus Lagrangian perspective (2D Euler).

flow map X

$$X : D \times \mathbb{R}^+ \to D$$

$u : D \to \mathbb{R}^2$ (domain)

$(\nabla \cdot u) = 0$

$\det(\nabla X) = 1$

$X^B = (X^F)^{-1}$

$$\left.\begin{array}{l} \partial_t \varphi + (u \cdot \nabla) \varphi = 0 \\ \varphi(x, t=0) = \varphi_0(x) \end{array}\right\} \text{method of characteristics}$$

Looking for: $X^B(\gamma(t), t) = \gamma_0$; for all curves $\gamma(t)$.
(pullback)?

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$u(x, t=0) = u_0(x)$,

$\boxed{\omega = \nabla \times u}$

$\partial_t \omega + (u \cdot \nabla)\omega = 0$

$\nabla \cdot u = 0$

$\boxed{u = -\Delta^{-1}(\nabla \times \omega)}$ Biot-Savart's law.

$$\boxed{D\omega_{(t,x)} = D\omega_0 \cdot DX(x,t)}$$

$D X_{(t,0]} = D X_{[\tau_1, 0]} \, D X_{[\tau_2, \tau_1]} \cdots$

3D Euler: $\boxed{\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u} + IC\,(\omega = \omega_0 u)$ Vortex stretching

Numerics: $(\partial_t u + u\cdot\nabla)\, x^{\beta} = \omega_0$

$\{\omega(t,x) = (\nabla x^{\beta})^{-1}\, \omega_0(x^{\beta})\}$

Where, $u = -\Delta^{-1}(\nabla\times\omega)$; $\det(\nabla x^{\beta}) = 1$.

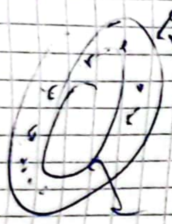
→ Numerics

→ $(\theta_1, \theta_2 \ldots \theta_m) \in T^m$

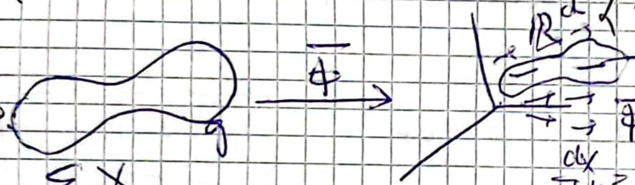
$\frac{d\theta_i}{dt} = \omega_i$ 1 dim

$\{\theta_1, \ldots, \theta_m\}$ T^m

x_3



$\{\omega_1, \ldots, \omega_m\} \in T^m$



Proof: Σ, X $\xrightarrow{\Phi}$ $\mathbb{R}^d$, $\{v_i\}_n$

$d\Phi(X)$

$\Phi(\Sigma)$

$\frac{dx}{dt} = B(x)$

$B(x) = d_{\Phi^{-1}(x)}\Phi(X)$.

$\boxed{\Delta\psi_i = \lambda_i \psi_i}$

For large N, this is an embedding

Lemma: $\Phi(p) = (\psi_1(p), \psi_2(p), \ldots, \psi_n(p))$

$\Sigma \to \mathbb{R}^n$.

$X(p) = x^{\mu}_{(p)} e_{\mu}(p)$

$$d_p \Phi(X) = \sum_{j=1}^{N} \sum_{\mu} \underbrace{X^{\mu}(p)\, \partial_{e_\mu} \psi_j(p)}\, \frac{\partial}{\partial x_j}.$$

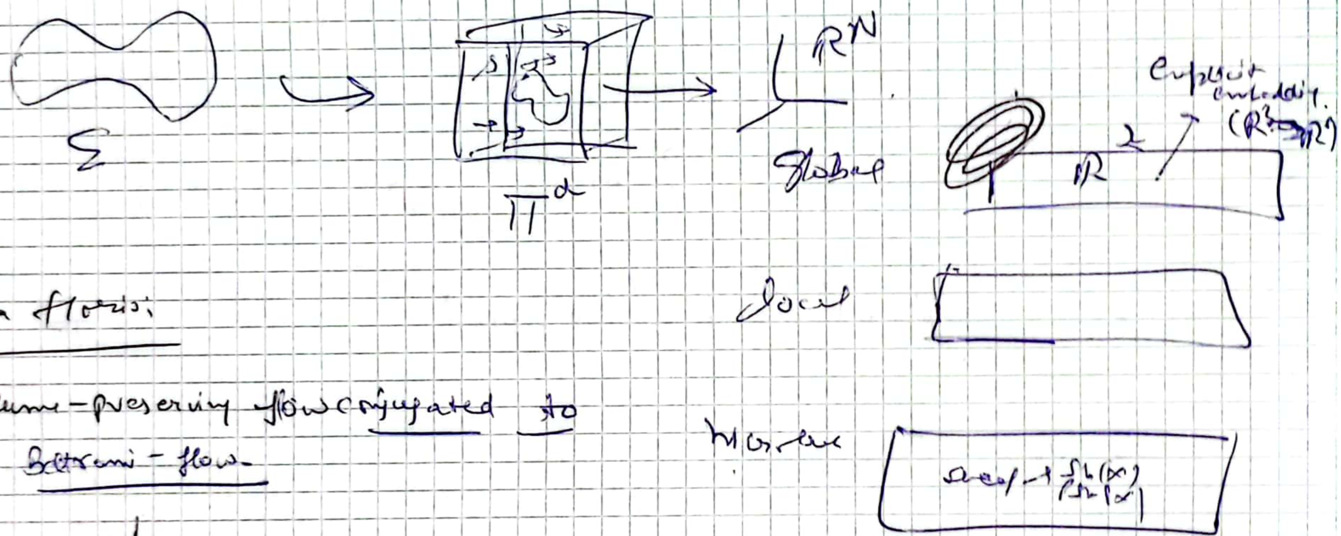
* Embedding(1,2)

$$X^{\mu}(p) \approx \sum_{\alpha=1}^{N} c_\alpha u_\alpha(p)$$

$$\left\{ \partial_{e_\mu} \psi_j(p) \approx \sum_{\ell=1}^{\infty} b_{j\ell} \psi_\ell(p) \right\}$$

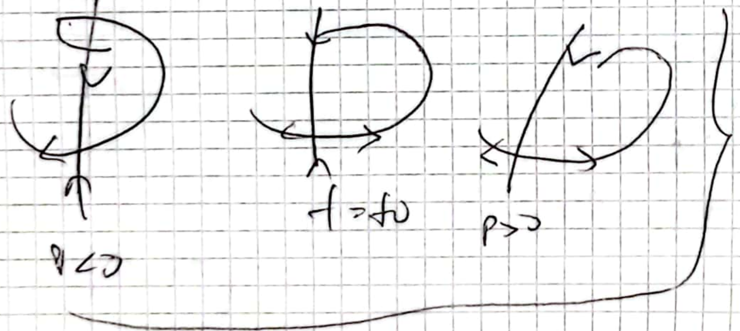
$$\frac{\partial}{\partial x_j} = \sum_{j=1}^{N} \sum_{k,\ell,\mu} a^{\mu}_{k} b_{\mu j \ell}\, \underbrace{\psi_k(p)}_{X_k}\, \underbrace{\psi_\ell(p)}_{u_\ell}$$

Classical embeddings:



→ Area flows:

Volume-preserving flow conjugated to a Beltrami-flow.



(* No blackboard, but just around me).

In this setting, ∃ horseshoe that has robust heteroclinic orbits

$H \curvearrowright M(S,M) \curvearrowleft G$

$\mathcal{D}iff(S)$

$X_f \in \mathfrak{X}_{ham}(M)$

$\# \quad 0 \to H^{m-2}_{dR}(M) \to \Omega^{m-2}(M) / d\Omega^{m-3}(M) \to \mathfrak{X}_{ex}(M) \to 0$

$Sp(2n) \curvearrowright M^{sk\,m}_{2n \times m}(\mathbb{R}) \curvearrowleft O(m)$

$gl(n)^* \xleftarrow{J_L} M_x \xrightarrow{J_R} gl(m)^*$

$\boxed{d\theta = ?}$

C^∞ frechet

$Emb(S,M)$

Sympl. form

Embedding

Idealfluid ↦ Dualpair

$(Emb(S,M), \Omega)$

$J_L \qquad J_R$

$C^d(M)^*$

$\mathfrak{X}_{vol}(S)^* = \Omega^1(S)/d\Omega^0(S)$

Non-exact symplectic

$\boxed{H^1(S) = 0}$

$H^{k-2}(S) \to \mathfrak{X}_{ex}(S) : \Omega^{k-2}(S) / d\Omega^{k-3}(S) \to \mathfrak{X}_{ex}(S)$

$$\partial_t v + \mathcal{L}_u v = -dp \quad (1\text{-form})$$

$$\partial_t \omega + \mathcal{L}_u \omega = 0 \quad (2\text{-form})$$

$$u = T_g \omega,$$

where, $T_g = -g^{-1} \cdot \Delta_g^{-1} \underline{\operatorname{div}_g}$.

(Laplace-Beltrami operator).

→ Momentum conservation

→ Helicity conservation (simple, modified).